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WAVES OVER BEACHES OF SMALL SLOPE, UNDER A DOCK, UNDER AN OVERHANGING CLIFF, AND PAST PLANE BARRIERS

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INTRODUCTION

During the past year a group working in the Institute for Mathematics and Mechanics, under Navy Contract N6ori-201, Task Order No. 1, has pursued theoretical studies of phenomena concerning surface gravity waves. Some of the results are being reported in a series of papers of which the present introductory report is the first. It contains a brief description of the papers which will follow. Moreover, it contains an exposition of the theory which is common to all of them in order to avoid needless repetition and to fix a common notation insofar as that is possible.

R. Courant

Waves over Beaches of Small Slope, Under a Dock, Under an Overhanging Cliff, and Past Plane Barriers

1. Brief Summary

The problem of obtaining two dimensional progressing waves over beaches with slope angle $\omega = \pi/2n$, n an integer, has been discussed recently by Bondi [1]*, Miche [4], Lewy [3], and Stoker [5]**. All of the solutions given by these authors become more complicated and cumbersome as n becomes larger, that is, as the beach slope becomes smaller. In fact, the solutions consist of finite sums of complex exponentials and exponential integrals, and the number of the terms in these sums increases with n. Actual ocean beaches usually slope rather gently, so that many of the interesting cases are just those in which the slope angle is small--of the order of a few degrees, say. K. O. Friedrichs has obtained a representation of the solution of the problem for all n in the form of a single complex integral, which can in turn be treated by the saddle point method to yield asymptotic solutions valid for large n, that is for beaches with small slopes. The resulting asymptotic representation turns out to be rather simple--even for numerical computations--and it appears to be very accurate. A comparison with the exact numerical solution for $\omega = 6^\circ$ (given in [5]) shows the asymptotic solution to be practically identical with the exact

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*Numbers in square brackets refer to the bibliography at the end of this report.

**The introduction to this paper describes all of these contributions (as well as others) in some detail.

solution all the way from infinity to within a distance of less than a wave length from the shore line.

H. Lewy [3] has obtained progressing wave solutions over beaches with slope angles $\omega = \frac{p\pi}{2n}$, with p any odd integer such that $p < 2n$. Thus the theory is available for cases in which ω is greater than $\pi/2$, so that the "beach" becomes an overhanging cliff. The solution for a special case of this kind, i.e. for $\omega = 135^\circ$ or $p = 3$, $n = 2$, has been carried out numerically by E. Isaacson in a report now being reproduced. It turns out that there is at least one interesting contrast with the solutions for waves over beaches in which $\omega < \pi/2$. In the latter case it has been found that as a progressing wave moves in toward shore the amplitude first decreases to a value below the value at ∞, before it increases and becomes very large at the shore line. The same thing holds for standing waves: At a certain distance from shore there exists always a crest which is lower than the crests at ∞. In the case of the overhanging cliff with $\omega = 135^\circ$, however, the reverse is found to be true: The first maximum going outward from the shore line is about 1 $^\circ/o$ higher than the height of the crests at ∞. Still another fact regarding the behavior of the solutions near the shore line is interesting. In all cases there exists just one standing wave solution which has a finite amplitude at the shore line; H. Lewy has shown that the ratio of the amplitude there to the amplitude at ∞ is given in terms of the angle ω by the formula $\sqrt{\pi/2\omega}$. Thus for angles ω less than $\pi/2$ the amplitude of the standing wave with finite amplitude is greater on shore than it is at infinity (becoming very large as ω becomes small) while for angles ω greater than $\pi/2$ the amplitude on shore is less than at ∞. This result has been verified by E. Isaacson for the

special case $\omega = 3\pi/4$. Since the observations indicate that the standing wave of finite amplitude is likely to be the wave which actually occurs in nature for angles ω greater than about 40°, the above results can be used to give a rational explanation for what might be called the "wine glass" effect: Wine is much more apt to spill over the edge of a glass with an edge which is flared outward than from a glass with an edge turned over slightly toward the inside of the glass.

A limit case of the problem of the overhanging cliff has a special interest, namely the case in which approaches the value π and the problem becomes what might be called the "dock problem": The water surface is free up to a certain point but from there on it is covered by a rigid horizontal plane. The solutions given by Lewy are so complicated as p and n become large that it seems hopeless to consider the limit of his solutions as $\omega \to \pi$. Friedrichs and Lewy have, however, attacked and solved the dock problem directly. Their results will appear shortly.

A fourth report by F. John deals with the effect of a plane rigid barrier on the surface waves in water which is, apart from the barrier, everywhere infinite in depth. The barrier is assumed to be inclined at an angle $\pi/2n$, n an integer, to the horizontal and to extend in the one case from the surface down into the water for a finite distance and in the other case to extend from infinity up to a point a certain finite distance below the water surface. The special case of a vertical barrier has been treated by F. Ursell [6] by a method different from that used by F. John, but which does not yield all possible solutions behaving like the classical steady progressing wave

solutions at infinity. One of the interesting results obtained by F. John is the following: Consider first the case of a plane barrier extending from a point below the water surface down to infinity. If it is prescribed that a progressing wave of fixed amplitude and frequency comes in from the right hand infinity, say, and that a progressing wave of the same frequency but unknown amplitude goes past the barrier to the left hand infinity while another is reflected back to the right hand infinity, it turns out that the problem has a uniquely determined solution provided that quite reasonable assumptions are made about the singularity at the tip of the barrier. In other words, a uniquely determined "reflection coefficient" for the barrier is obtained in this case simply by stipulating that there should be no progressing wave coming toward the barrier from the left hand infinity. However, in the case of a barrier extending from the surface downward a finite distance the situation is quite different in this respect. No uniquely defined reflection coefficient can be determined because of the fact that the strength of the singularity at the intersection of the barrier with the water surface cannot be determined a priori. This fact was observed earlier in dealing with a similar problem for waves on sloping beaches; the fact is that one would probably have to utilize knowledge of some sort about the phenomena of breaking of waves--which are essentially non-linear phenomena and hence out of the scope of the theory in question here--in order to determine reflection coefficients in these cases without recourse to experimental results. The results of observations (cf. the paper of Miche [4]) on sloping beaches seem to indicate

the following to be likely: If a barrier slopes at a small angle then practically all the energy of a progressing wave moving in the direction of decreasing depth over the barrier will be converted either into heat or into the energy of a flow (the undertow) through the occurrance of breakers. If, however, the wave amplitude is very small or the slope angle is large enough (greater than 40°, according to Miche) a standing wave, denoting perfect reflection, will be observed over the barrier in practice.

2. Resumé of the general linear theory of surface waves of small amplitude

In this and the following section we summarize the mathematical theory common to the papers discussed above. At the same time the notations to be used will be fixed as far as that is practicable.

In all of the papers mentioned above it is assumed that the motion is a two-dimensional irrotational motion of an incompressible perfect fluid subject to gravity. It follows that there exists a potential function $\Phi(x,y;t)$ depending on the time t as well as on the two space coordinates x,y (since we wish to deal with unsteady motions), from which the velocity components (u,v) are obtained as the derivatives (Φ_x, Φ_y).*

We proceed to formulate completely a typical problem for the determination of Φ .** Fig. 2.1 indicates the case of a uniformly sloping beach with the origin at the

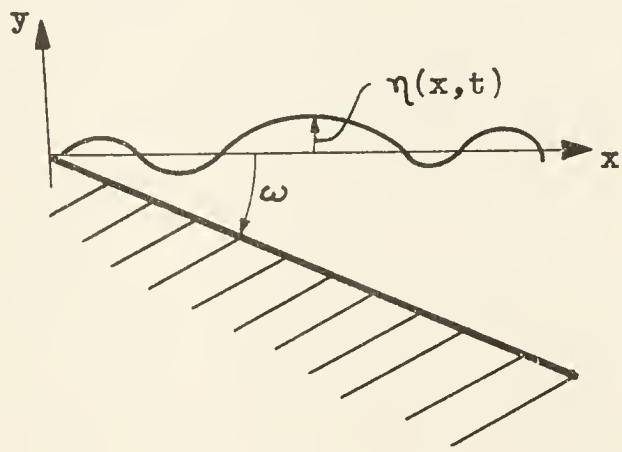

Fig. 2.1

*Subscripts x,y,t always mean differentiations with respect to these quantities.

**See any standard treatise, Lamb [2] for example.

beach, the x-axis in the undisturbed water surface, and the y-axis positive upward. The potential function Φ must satisfy the Laplace equation

$$\Phi_{xx} + \Phi_{yy} = 0 \tag{2.1}$$

at all points in the interior of the fluid. At the displaced free surface (which we denote by $y = \eta(x,t)$) the conditions to be satisfied are the dynamical condition that the pressure should be constant (we take it to be zero) and the kinematical condition that the free surface should consist always of the same particles. By making use of Bernoulli's law and also dropping all quadratic terms in Φ, η, and their derivatives on the assumption that all such quantities are so small that powers of them above the first can be neglected,* one finds without difficulty that Φ must satisfy the following free surface condition

$$\Phi_{tt} + g\Phi_y = 0 \quad \text{at} \quad y = 0. \tag{2.2}$$

(It is consistent with the above linearization to satisfy the free surface condition at its original undisturbed position $y = 0$ rather than at $y = \eta(x,t)$.) At the bottom the condition to be satisfied is

$$\Phi_n = 0 , \tag{2.3}$$

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*For a discussion of the exact theory and other types of approximations to it which do not assume small velocities and displacements, see [5].

which states that the velocity of the water is tangential to the bottom. In addition to the differential equation and the boundary conditions it is necessary to prescribe conditions at the origin (i.e. at the shore line) and at ∞, if Φ is to be uniquely determined; these conditions are formulated a little later.

Since the motions we consider are unsteady motions, it would also be appropriate and necessary in general to prescribe initial conditions at the time $t = 0$. However, in all that follows we consider only motions which are simple harmonic in the time, so that we may write

$$\Phi(x,y;t) = e^{i\sigma t}\varphi(x,y) \tag{2.4}$$

with the understanding that either the real or the imaginary part of the right hand side is to be taken. The conditions (2.1), (2.2), and (2.3) are at once seen to take the forms

$$\varphi_{xx} + \varphi_{yy} = 0 , \tag{2.5}$$

$$\varphi_y - \frac{\sigma^2}{g}\varphi = 0 \quad \text{at } y = 0 , \tag{2.6}$$

$$\varphi_n = 0 \quad \text{at the bottom,} \tag{2.7}$$

in terms of the potential function $\varphi(x,y)$ depending on x and y only. The surface elevation η is given by

$$\eta = \frac{1}{g}\Phi_t\Big|_{y=0} = \frac{i\sigma}{g}e^{i\sigma t}\varphi\Big|_{y=0} . \tag{2.8}$$

Thus the surface displacement η is proportional to the surface value of φ .

It is interesting and useful for the sake of later developments to give the well known solutions $\varphi(x,y)$ for the special case of water infinite in depth everywhere. The functions

$$(2.9) \qquad \varphi = Ae^{my}\cos(mx + \alpha) \; ; \quad A, m, \alpha \text{ arbitrary}$$

are solutions of (2.5), which will in addition satisfy the free surface condition (2.6) provided that the relation

$$(2.10) \qquad \sigma^2 = gm = g\,\frac{2\pi}{\lambda}$$

between the "frequency" σ and the wave length λ is satisfied. In other words, upon reintroduction of the "time factor" $e^{i\sigma t}$, standing wave solutions are found in the form of simple sine or cosine functions of x; these solutions die out exponentially in the depth.*

By combining two of the standing wave solutions appropriately one obtains progressing wave solutions of the type

$$(2.11) \qquad \Phi(x,y;t) = Ae^{my}\cos(mx \pm \sigma t + \alpha) \; ,$$

that is, steady simple harmonic waves moving in either

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*It has been shown (cf. [5], p. 7) that these solutions for which φ and the "velocity components" φ_x and φ_y are bounded at ∞ are the only ones. It is not necessary to require that the solutions behave like sines or cosines in x at ∞--the boundedness requirement is sufficient.

direction along the x-axis. The phase or propagation speed c of these waves is given by

$$c = \sigma/m = \sqrt{\frac{g}{m}} = \sqrt{\frac{g\lambda}{2\pi}}\ , \tag{2.12}$$

that is, the propagation speed is not independent of the wave length λ , but is proportional to its square root. The wave motion is thus dispersive in character.

In case the depth of the water is not infinite but has a constant value h, progressing wave solutions are also easily obtained, as follows:

$$\Phi(x,y;t) = A \cosh my \cos(mx \pm \sigma t + \alpha)\ . \tag{2.13}$$

However, the relations between σ and $\lambda = 2\pi/m$ and between c and λ are different. They are now

$$\sigma^2 = gm \tanh mh \tag{2.14}$$

and

$$c = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}}\ . \tag{2.15}$$

If h is very large compared with λ we note that c is practically given by (2.12), but if h is small compared with λ we may replace $\tanh \frac{2\pi h}{\lambda}$ by $\frac{2\pi h}{\lambda}$ and c is then given approximately by

$$c \simeq \sqrt{gh}\ . \tag{2.16}$$

In other words, in water that is very shallow compared with

the wave length the propagation speed depends on the square root of the depth and is nearly independent of the wave length.

It is convenient for later purposes to introduce the following dimensionless variables replacing x,y, and t:

$$x_1 = mx \ , \ y_1 = my \ , \ t_1 = \sigma t \ , \tag{2.17}$$

with m defined in terms of σ by (2.10). This means that the new unit of length is proportional to the wave length of a steady wave of frequency σ in water of finite depth. In terms of these variables φ of course continues to be a potential function. The free surface condition (2.6) takes the form

$$\varphi_{y_1} - \varphi = 0 \quad \text{at } y_1 = 0 \ . \tag{2.18}$$

From now on, in this report as well as in subsequent reports, we shall make use of the new independent variables but <u>we shall omit the subscript</u> unless the contrary is explicitly stated.

3. Progressing waves over sloping beaches

In this section we give a brief outline of the method and results of Lewy and Stoker for the progressing wave solutions over sloping beaches when the beach slopes at an angle $\omega = \pi/2n$ (cf. Fig. 2.1). This will serve as a contrast to the simpler representation of these solutions to be given in a subsequent report by Friedrichs and also will provide some of the main ideas used by F. John to obtain the solutions for the case of plane barriers inclined at the angles $\pi/2n$.

We introduce the complex potential function $\chi(z)$ of the complex variable $z = x + iy = re^{i\theta}$:

$$\chi(z) = \varphi + i\Psi , \tag{3.1}$$

in which φ is the real potential function introduced above. Thus $\chi(z)$ is analytic in the sector $-\omega \leq \theta \leq 0$. We may write

$$\left(\frac{\partial}{\partial y} - 1\right)\varphi = \operatorname{Im} i\left(\frac{\partial}{\partial y} - 1\right)(\varphi + i\Psi) = \operatorname{Im}\left(-\frac{d}{dz} - i\right)\chi(z)$$

so that the free surface condition (2.18) can be expressed in terms of χ as follows

$$\operatorname{Im}\left(\frac{d}{dz} + i\right)\chi(z) = 0 \quad \text{for} \quad \theta = 0 . \tag{3.2}$$

At the bottom the boundary condition is

$$\operatorname{Im} e^{-i\pi/2n}\frac{d}{dz}\chi(z) = 0 \quad \text{for} \quad \theta = -\frac{\pi}{2n} = -\omega . \tag{3.3}$$

The essential step in the method employed by Lewy and Stoker is the derivation of an ordinary non-homogeneous

differential equation with constant coefficients for the analytic function $\chi(z)$. The differential equation is

$$(3.4) \qquad D^n \cdot (D - \zeta_1)(D - \zeta_2) \ldots (D - \zeta_n)\ \chi(z) = F(z)\ ,$$

in which D is the operator $\frac{d}{dz}$ and the ζ_k are the unit complex numbers

$$(3.5) \qquad \zeta_k = ie^{i\pi k/n}, \quad k = 1, 2, \ldots, n\ .$$

The analytic function F(z) is determined through the properties of $\chi(z)$, which in turn result from those postulated for $\varphi(x,y)$. The chain of reasoning is briefly the following: On account of (3.2) and (3.3) and the special form of the differential operator on the left side of (3.4) it is found that Im F(z) <u>vanishes on both boundaries of the sector</u>--this in fact is the property which fixed the choice of the differential operator--and therefore F(z) can be continued analytically by successive reflections into a function which is singlevalued in the entire plane. Here we use the fact that $\omega = \pi/2n$ with n an integer. If now we assume φ as well as its derivatives up to a certain order to be bounded in the entire sector, so that $\chi(z)$ would have a similar property, it follows almost immediately from Liouville's theorem that $F(z) \equiv 0$. It turns out then that one can obtain a solution of the resulting homogeneous equation for $\chi(z)$ (in the form of a sum of complex exponentials) which satisfies the boundary conditions; we call this solution $\chi_1(z)$. Then the real potential function $\varphi_1(x,y) = \mathrm{Re}\ \chi_1(z)$ is found to behave for large values of x like

$$e^y \cos\left(x + \frac{n-1}{4}\pi\right)$$

so that only a particular type of standing wave solution with a unique phase at ∞ is obtained upon reintroduction of the time factor, and no progressing wave solutions can be constructed. In order to obtain standing wave solutions which are "out of phase at ∞" with the solutions $\varphi_1(x,y)$ it is therefore essential to admit a singularity. It is, on the other hand, natural on various grounds (cf. [5], p. 11) to admit a logarithmic singularity of $\varphi(x,y)$ at the origin; if one does it is then readily seen (once more by applying Liouville's theorem) that F(z) must be the function

$$F(z) = A\,\frac{1}{z^{2n}}\,, \tag{3.6}$$

with A an arbitrary real constant. If A = 0 we obtain the solution $\chi_1(z)$ already mentioned; if $A \neq 0$ a solution $\chi_2(z)$ of (3.4) with F(z) given by (3.6) can be found which satisfies the boundary conditions (3.2) and (3.3) and for which $\varphi_2(x,y) = \mathrm{Re}\,\chi_2(z)$ behaves for large x like

$$e^y \sin\left(x + \frac{n-1}{4}\pi\right) \quad .$$

Thus a second standing wave solution out of phase at ∞ with φ_1 is determined and progressing wave solutions can be constructed.

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